

The cohomological reduction method for computing n -dimensional cocyclic matrices

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Abstract

Provided that a cohomological model for G is known, we describe a method for constructing a basis for n -cocycles over G , from which the whole set of n -dimensional cocyclic matrices over G may be straightforwardly calculated. Focusing in the case $n = 2$ (which is of special interest, e.g. for looking for cocyclic Hadamard matrices), our method provides a basis for 2-cocycles in such a way that representative 2-cocycles are calculated all at once, so that there is no need to distinguish between inflation and transgression 2-cocycles (as it has traditionally been the case until now). When $n > 2$, this method provide an uniform way of looking for higher dimensional *cocyclic* Hadamard matrices for the first time. We illustrate the method with some examples, for $n = 2, 3$. In particular, we give some examples of improper 3-dimensional cocyclic Hadamard matrices.

Key words: (co)homological model, integer Smith normal form, cocyclic matrix, proper/improper higher dimensional Hadamard matrices

1. Introduction

A Hadamard matrix H is a $n \times n$ square matrix with entries in $\{1, -1\}$ such that $H \cdot H^T = nI$. It is straightforward to check that a necessary condition for the existence of such a matrix is that $n = 1, 2$ or n is a multiple of 4. It is conjectured that this condition is sufficient as well, but it has not been proved yet. The interested reader is referred to (15) for further information on Hadamard matrices and their applications.

Since the introduction of cocyclic Hadamard matrices in (17), the cocyclic framework has been shown to be a good choice for dealing with the Hadamard Conjecture about the

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existence of Hadamard matrices of order $4t$, $t \in \mathbf{N}$. In particular, dihedral groups D_{4t} and abelian groups $\mathbf{Z}_t \times \mathbf{Z}_2^2$ might provide a large amount of cocyclic Hadamard matrices for every $t \in \mathbf{N}$ (see (10), (12) and (1) for details).

As introduced in (18; 19), the notion of higher dimensional Hadamard matrix is a natural generalization of usual Hadamard matrices to the case of n -dimensional matrices, for $n \geq 2$.

One could wonder whether the cocyclic framework could also be taken into account for higher dimensional Hadamard matrices. The main difficulty here is determining the set of n -cocyclic matrices over a given finite group G . Though there are some methods available for constructing a basis for 2-cocyclic matrices over G , we have no evidence of the existence of a general method for calculating the set of n -cocyclic matrices over G .

Focusing in the case $n = 2$ (which is of special interest, e.g. for looking for cocyclic Hadamard matrices), the Universal Coefficient Theorem provides a decomposition of representative 2-cocycles over a group G as the direct sum of the inflation and transgression cocycles over G ,

$$H^2(G, \mathbf{Z}_2) \cong \text{Ext}(G/[G, G], \mathbf{Z}_2) \oplus \text{Hom}(H_2(G), \mathbf{Z}_2).$$

Until now, all the methods which look for a basis for 2-cocycles uses this decomposition, so that two different processes have to be performed.

In spite of this fact, there is a chance to calculate a basis for $H^2(G, \mathbf{Z}_2)$ all at once in a straightforward manner, provided that a cohomological model for G is known, so that there is no need to distinguish between inflation and transgression 2-cocycles, as it has traditionally been the case. It is the cohomological analog of the homological reduction method described in (7). An early precedent of this technique is located in (13), focusing on p -groups.

This procedure may be extended in order to construct a basis for n -cocycles as well as n -dimensional cocyclic matrices over a group G for which a cohomological model $\text{coh}G$ is known. We call this process the ‘‘cohomological reduction method’’.

The term *cohomological model* refers to a special type of homotopy equivalence (termed *contraction* (11))

$$\phi: \text{Hom}(\bar{B}(\mathbf{Z}[G]), \mathbf{Z}_2) \xrightleftharpoons[K]{F} \text{coh}G$$

from the set of homomorphisms of the *reduced bar construction* (i.e. the reduced complex associated to the standard bar resolution (11)) of the group G onto \mathbf{Z} , to a differential graded comodule of finite type $\text{coh}G$. Thus

$$H^*(G) = H^*(\text{Hom}(\bar{B}(\mathbf{Z}[G]), \mathbf{Z}_2)) = H^*(\text{coh}G)$$

and the n -cohomology of G and its representative n -cocycles may be effectively computed from those of $\text{coh}G$, by means of Veblen’s algorithm (20) (involving the Smith’s normal forms of the matrices representing the codifferential operator).

In particular, if $\phi: \bar{B}(\mathbf{Z}[G]) \xrightleftharpoons[K]{F} (hG, d)$ defines a homological model for G , then

$$\phi^*: \text{Hom}(\bar{B}(\mathbf{Z}[G]), \mathbf{Z}_2) \xrightleftharpoons[F^*]{K^*} (\text{Hom}(hG, \mathbf{Z}_2), d^*)$$

defines a cohomological model for G . This way, the set of groups for which a cohomological model is known includes those groups for which a homological model is known. Consequently, this method extends the homological reduction method described in (7).

Comparison with other methods already available for 2-cocycles shows that the cohomological reduction method often reduces both the number and size of the operations involved, excepting the case of the familiar homological reduction method of (8). In fact, the calculation of 2-cocycles coming from inflation involves the differential operator d_2 and the projection F_1 , whereas the calculation of 2-cocycles coming from transgression involves d_3 and F_2 . Accordingly, d_2 is needed for calculating $Im\ d^1$ (in fact, $d^1 = d_2^*$), while d_3 is needed for calculating $Ker\ d^2$ (in fact, $d^2 = d_3^*$) and F_2^* constitutes the injection homomorphism from $cohG^2$ to $Hom(\bar{B}(\mathbf{Z}[G]), \mathbf{Z}_2)$. Thus the same information is required in the two procedures, though the cohomological reduction method is somehow more elegant than the homological reduction one (see the examples of Section 3 below). Nevertheless, at this point, it is noticeable that the cohomological reduction method provides a full basis for n -cocycles over G , for $n > 2$, whereas the former homological reduction method of (7) provides a basis just for 2-cocycles.

We organize the paper as follows. In section 2 we describe the cohomological reduction method itself, that is, how to construct a full basis for n -cocycles over G from a cohomological model $cohG$ for G . Section 3 is devoted to show several 2 and 3 dimensional examples, including the well-known cases of dihedral groups D_{4t} and abelian groups $\mathbf{Z}_{2t} \times \mathbf{Z}_2$ and $\mathbf{Z}_t \times \mathbf{Z}_2^2$ for clarity. All the calculations have been made with aid of some packages in Mathematica provided by the authors in (4; 5; 6). The last section is devoted to conclusions and related works, such as the application of the method for constructing improper higher dimensional cocyclic Hadamard matrices.

2. Describing the cohomological reduction method

The cohomological reduction method provides a computationally efficient way to lift the cohomological information from a cohomological model

$$\phi: Hom(\bar{B}(\mathbf{Z}[G]), \mathbf{Z}_2) \xrightleftharpoons[K]{F} cohG$$

for a group G to the group itself. The injection morphism

$$K : cohG \longrightarrow Hom(\bar{B}(\mathbf{Z}[G]), \mathbf{Z}_2)$$

helps in this task.

Let $\mathcal{B}_{i-1} = \{\mathbf{u}_1, \dots, \mathbf{u}_q\}$, $\mathcal{B}_i = \{\mathbf{e}_1, \dots, \mathbf{e}_r\}$ and $\mathcal{B}_{i+1} = \{\mathbf{v}_1, \dots, \mathbf{v}_s\}$ be the corresponding basis for $cohG$ at dimensions $i-1$, i and $i+1$, respectively. Attending to Veblen's algorithm, since $H^i(G; \mathbf{Z}_2) \cong H^i(cohG; \mathbf{Z}_2) = Ker\ d^i / Im\ d^{i-1}$, we need to calculate the binary (i.e. with coefficients in \mathbf{Z}_2) Smith Normal Forms of the matrices representing the codifferential operators d^{i-1} and d^i ,

$$M_{i-1}(d) = \begin{pmatrix} d(\mathbf{u}_1) \\ \vdots \\ d(\mathbf{u}_q) \end{pmatrix}_{q \times r} \quad D_{i-1} = \left(\begin{array}{c|c} I_l & 0 \\ \hline 0 & 0 \end{array} \right)_{q \times r} \quad M_i(d) = \begin{pmatrix} d(\mathbf{e}_1) \\ \vdots \\ d(\mathbf{e}_r) \end{pmatrix}_{r \times s} \quad D_i = \left(\begin{array}{c|c} I_k & 0 \\ \hline 0 & 0 \end{array} \right)_{r \times s}$$

so that $H^i(G; \mathbf{Z}_2) \cong H^i(cohG; \mathbf{Z}_2) \cong \mathbf{Z}_2^{r-k-l}$.

Furthermore, some change basis matrices P_j and Q_j exist, for $j = i - 1, i$, such that

$$\begin{array}{ccc} \mathcal{B}_j & \xrightarrow{M_j(d)} & \mathcal{B}_{j+1} \\ P_j \uparrow & \# & \downarrow Q_j \\ \bar{\mathcal{B}}_j & \xrightarrow{D_j} & \bar{\mathcal{B}}_{j+1} \end{array} \quad D_j = P_j \cdot M_j(d) \cdot Q_j$$

Now we proceed according to the following steps:

- (1) A basis $\bar{\mathcal{C}}$ for $\text{Im } d^{i-1}$ with regards to \mathcal{B}_i is obtained from the first l columns of Q_{i-1}^{-1} . Thus $\bar{\mathcal{C}}$ is a basis for i -coboundaries over $\text{coh}G$.
- (2) A basis $\bar{\mathcal{D}}$ for $\text{Ker } d^i$ with regards to \mathcal{B}_i is obtained from the $r - k$ last rows of P_i . Thus $\bar{\mathcal{D}}$ is a basis for i -cocycles over $\text{coh}G$.
- (3) Select those $r - k - l$ elements in $\bar{\mathcal{D}}$ which are not linear combinations of the elements in $\bar{\mathcal{C}}$. This conforms a basis $\bar{\mathcal{B}}$ for representative i -cocycles over $\text{coh}G$.
- (4) Lift $\bar{\mathcal{B}}$ to the correspondent basis \mathcal{B} in $H^i(\bar{B}_i(\mathbf{Z}[G]); \mathbf{Z}_2)$ by means of the injection K .

Graphically,

$$\begin{array}{ccc} \text{Hom}(\bar{B}_i(\mathbf{Z}[G]), \mathbf{Z}_2) & \xleftarrow{K} & \mathcal{B}_i \\ & & \uparrow P_i, Q_{i-1}^{-1} \\ & & \bar{\mathcal{B}}_i \end{array}$$

Proposition 1. *The scheme above defines a basis \mathcal{B} for representative i -cocycles over G .*

A basis for i -coboundaries may be obtained by Linear Algebra. More concretely, denote $\partial_{[g_1, \dots, g_{i-1}]} : \bar{B}_i(\mathbf{Z}[G]) \rightarrow \mathbf{Z}_2$ the i -coboundary associated to the characteristic map $\delta_{[g_1, \dots, g_{i-1}]}$ of the element $[g_1, \dots, g_{i-1}] \in B_{i-1}(\mathbf{Z}[G])$,

$$\partial_{[g_1, \dots, g_{i-1}]}([h_1, \dots, h_i]) = \delta_{[g_1, \dots, g_{i-1}]}([h_2, \dots, h_i]) + \delta_{[g_1, \dots, g_{i-1}]}([h_1, \dots, h_{i-1}]) +$$

$$\sum_{j=1}^{i-1} \delta_{[g_1, \dots, g_{i-1}]}([h_1, \dots, h_j h_{j+1}, \dots, h_i]) \quad \text{mod } 2$$

Take the $4t \times \dots \times 4t$ i -dimensional matrix $M_{\partial_{[g_1, \dots, g_{i-1}]}}$ related to $\partial_{[g_1, \dots, g_{i-1}]}$ as vectors of length $4^i t^i$. Moreover, consider the $4^{i-1} t^{i-1} \times 4^i t^i$ matrix C whose rows are the vectors $M_{\partial_{[g_1, \dots, g_{i-1}]}}$. Then a row reduction on C leads to a basis for i -coboundaries. It suffices to keep trace of those coboundaries $\partial_{[g_1, \dots, g_{i-1}]}$ whose transformed rows in $M_{\partial_{[g_1, \dots, g_{i-1}]}}$ after the row reduction are not zero.

Lemma 2. *The morphisms $\partial_{[g_1, \dots, g_{i-1}]}$ above define a basis for i -coboundaries.*

The cohomological reduction method provides then the following algorithm for computing n -dimensional cocyclic matrices over G .

Algorithm 1. (cohomological reduction method)

INPUT: group with cohomological model $\{G, \text{coh}G, F, K, \phi\}$

Construct a basis \mathcal{E} for i -coboundaries (Lemma 2).

Construct a basis \mathcal{B} for representative i -cocycles (Proposition 1).

OUTPUT: By juxtaposition, a basis $\mathcal{B} \cup \mathcal{E}$ for i -cocycles over G .

3. Examples

We next show how the cohomological reduction method works for constructing basis for 2-cocycles and 3-cocycles over some groups (which have been shown to provide many 2-cocyclic Hadamard matrices, see (8)).

All the executions and examples of this section have been worked out with aid of the *Mathematica 4.0* notebooks (4; 5) described in (3; 9) (for constructing homological models) and (2) (in order to form a basis for 2-cocycles from which the search for cocyclic Hadamard matrices is then developed), running on a *Pentium IV 2.400 Mhz DIMM DDR266 512 MB*.

In the sequel, the elements of a product $A \times B$ are ordered as the rows of a matrix indexed in $|A| \times |B|$. For instance, if $|A| = r$ and $|B| = c$, the ordering is

$$\langle a_1b_1, a_1b_2, \dots, a_1b_c, a_2b_1, a_2b_2, \dots, a_2b_c, \dots, a_rb_1, \dots, a_rb_c \rangle$$

The elements in the group are labeled from 1 to $|G|$, accordingly to this ordering.

Let consider the families of groups below (assume $\mathbf{Z}_k = \{0, 1, \dots, k-1\}$ with additive law).

- (1) $G_1^t = \mathbf{Z}_{2t} \times \mathbf{Z}_2$.
- (2) $G_2^t = \mathbf{Z}_t \times \mathbf{Z}_2^2 = \mathbf{Z}_t \times (\mathbf{Z}_2 \times \mathbf{Z}_2)$. Notice that $G_2^t \simeq G_1^t$ for odd t .
- (3) $G_3^t = D_{4t} = \mathbf{Z}_2 \rtimes_{\chi} \mathbf{Z}_{2t}$, $\chi : \mathbf{Z}_2 \times \mathbf{Z}_{2t} \rightarrow \mathbf{Z}_{2t}$ such that $\chi(1, x) = -x$ and $\chi(0, x) = x$.
Notice that $G_3^1 \cong G_2^1 \cong G_1^1 = \mathbf{Z}_2 \times \mathbf{Z}_2$ is abelian, but G_3^t is not abelian, for $t > 1$.

In this section we will construct a cohomological model for G_i^t from the homological models for G_i^t described in (8), so that if $\phi : \bar{B}(\mathbf{Z}[G_i^t]) \xrightarrow[F^*]{F} (hG_i^t, d)$ defines a homological model for G_i^t , then

$$\phi^* : Hom(\bar{B}(\mathbf{Z}[G_i^t]), \mathbf{Z}_2) \xrightarrow[F^*]{K^*} (Hom(hG_i^t, \mathbf{Z}_2), d^*)$$

defines a cohomological model for G_i^t . Here, as usual, we use $-^*$ for noting the dual object for $-$.

More concretely, it suffices to take duals on the basis \mathcal{B}_k for hG_i^t on degree $1 \leq k \leq 4$, and the differential operators $d_j : B_{j+1} \rightarrow B_j$ and the projections $F_j : \bar{B}_j(\mathbf{Z}[G_i^t]) \rightarrow B_j$ for $2 \leq j \leq 3$. Recall that

$$\bar{B}_k(\mathbf{Z}[G]) = \langle [g_1, \dots, g_k] : g_j \in G \rangle.$$

Notice that the matrices P and Q involved in the calculation of the Smith Normal Form, D , for a matrix A (so that $D = P \cdot A \cdot Q$) are not uniquely determined, in general. In the sequel we will use the matrices coming from the `SmithNormalForm` package programmed in (6).

3.1. Constructing basis for 2-cocycles

In the sequel, we use the following notation. The back negacyclic matrix of order j is denoted by $BN_j = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & & \ddots & -1 \\ \vdots & \ddots & \ddots & \vdots \\ 1 & -1 & \cdots & -1 \end{pmatrix}_{j \times j}$, as usual. The square matrix of order n formed all of 1s is denoted by 1_n . The Kronecker product of matrices is denoted by \otimes , so that $A \otimes B$ is the block matrix $\begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & & \vdots \\ a_{n1}B & \cdots & a_{nn}B \end{pmatrix}$. The Hadamard (pointwise) product of matrices is simply denoted as $A \cdot B$. We use the Kronecker-Iverson notation $[b]$ (see (14)), which evaluates to 1 for Boolean expressions b having value true, and to 0 for those having value false. Finally, the notation $[x]_m$ refers to $x \bmod m$.

3.1.1. Basis for $G_1^t = \mathbf{Z}_{2t} \times \mathbf{Z}_2$

Notice that the i -th element of G_1 corresponds to $(\lfloor \frac{i-1}{2} \rfloor, [i-1]_2) \in \mathbf{Z}_{2t} \times \mathbf{Z}_2$. Conversely, the element $(i_1, i_2) \in \mathbf{Z}_{2t} \times \mathbf{Z}_2$ corresponds to the i -th element of G_1 , for $i = 2i_1 + i_2 + 1$.

In these circumstances, it may be checked that $\mathcal{B}_1^* = \{u_1^*, u_2^*\}$, $\mathcal{B}_2^* = \{e_1^*, e_2^*, e_3^*\}$, $\mathcal{B}_3^* = \{v_1^*, v_2^*, v_3^*, v_4^*\}$, $d^1 = 0$, $d^2 = 0$, $F[i|j] = [[i_1 + j_1 \geq 2t] \cdot e_1 + i_1 j_2 \cdot e_2 + [i_2 + j_2 \geq 2] \cdot e_3]_2$.

From these data, it may be checked that

D_1	Q_1^{-1}	D_2	P_2
$0_{2 \times 3}$	I_3	$0_{3 \times 4}$	I_3

Thus $H^2(\text{coh}G_1^t; \mathbf{Z}_2) = \mathbf{Z}_2^3$ and $\mathcal{B}_2^* = \{e_1^*, e_2^*, e_3^*\}$ is a basis for representative 2-cocycles over $\text{coh}G_1^t$. Accordingly, a basis for representative 2-cocycles over G_1^t is given

$$\text{by } \{F^*(e_1^*), F^*(e_2^*), F^*(e_3^*)\} = \{BN_{2t} \otimes 1_2, 1_t \otimes \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & - & 1 & - \\ 1 & - & 1 & - \end{pmatrix}, 1_{2t} \otimes BN_2\}.$$

This basis may be extended to a full basis for 2-cocycles over G_1^t , by simply juxtaposing a basis for 2-coboundaries over G_1^t (see Lemma 2). This basis is $\langle \partial_2, \dots, \partial_{4t-2} \rangle$ as it was pointed out in (8).

Remark 3. A basis $\{\beta_1, \beta_2, \gamma_1\}$ for representative 2-cocycles over G_1^t was already determined in (8) (notice that G_1^t was denoted G_2^t there). It may be checked that $\beta_1 = F^*(e_3^*)$

and $\gamma_1 = F^*(e_2^*)$. If $2t = 2^r q$, for odd q , then

$$\beta_2 = 1_q \otimes BN_{2^r} \otimes 1_2 = F^*(e_1^*) \cdot \prod_{k=0}^{\lfloor \frac{q}{2} \rfloor - 1} \partial_{k2^{r+2}+2^{r+1}+1} \dots \partial_{k2^{r+2}+2^{r+2}}.$$

3.1.2. *Basis for $G_2^t = \mathbf{Z}_t \times \mathbf{Z}_2^2 = \mathbf{Z}_t \times (\mathbf{Z}_2 \times \mathbf{Z}_2)$*

Notice that the i -th element of G_2 corresponds to $(\lfloor \frac{i-1}{4} \rfloor, [\lfloor \frac{i-1}{2} \rfloor]_2, [i-1]_2) \in \mathbf{Z}_t \times (\mathbf{Z}_2 \times \mathbf{Z}_2)$. Conversely, the element $(i_1, i_2, i_3) \in \mathbf{Z}_t \times (\mathbf{Z}_2 \times \mathbf{Z}_2)$ corresponds to the i -th element of G_2 , for $i = 4i_1 + 2i_2 + i_3 + 1$.

In these circumstances, it may be checked that

$$\mathcal{B}_1^* = \{u_1^*, u_2^*, u_3^*\}, \mathcal{B}_2^* = \{e_1^*, \dots, e_6^*\}, \mathcal{B}_3^* = \{v_1^*, \dots, v_{10}^*\},$$

$$\begin{aligned} d^1(u_1^*) &= t \cdot e_1^*, d^2(e_2^*) = t \cdot v_2^*, d^2(e_3^*) = t \cdot v_3^*, F[i|j] = \\ &= [[i_1 + j_2 \geq t] \cdot e_1 + i_1 j_2 \cdot e_2 + i_1 j_3 \cdot e_3 + [i_2 + j_2 \geq 2] \cdot e_4 + i_2 j_3 \cdot e_5 + [i_3 + j_3 \geq 2] \cdot e_6]_2. \end{aligned}$$

From these data, it may be checked that:

t	D_1	Q_1^{-1}	D_2	P_2
$[t]_2 = 0$	$0_{3 \times 6}$	I_6	$0_{6 \times 10}$	I_6
$[t]_2 = 1$	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	I_6	$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

Thus $H^2(\text{coh}G_2^t; \mathbf{Z}_2) = \begin{cases} \mathbf{Z}_2^3 & \text{for } t \text{ odd} \\ \mathbf{Z}_2^6 & \text{for } t \text{ even} \end{cases}$ and $\begin{cases} \{e_4^*, e_5^*, e_6^*\} & \text{for } t \text{ odd} \\ \{e_1^*, e_2^*, e_3^*, e_4^*, e_5^*, e_6^*\} & \text{for } t \text{ even} \end{cases}$ is a basis for representative 2-cocycles over $\text{coh}G_2^t$. Accordingly, a basis for representative 2-cocycles over G_2^t consists in the correspondent $F^*(e_i^*)$, where $F^*(e_1^*) = BN_t \otimes 1_4$, $F^*(e_2^*) = 1_{\frac{t}{2}} \otimes K_2$, $F^*(e_3^*) = 1_{\frac{t}{2}} \otimes K_3$, $F^*(e_4^*) = 1_t \otimes BN_2 \otimes 1_2$, $F^*(e_5^*) = 1_t \otimes K_1$ and $F^*(e_6^*) = 1_{2t} \otimes BN_2$.

The matrices K_1, K_2, K_3 are given by

K_1	K_2	K_3
$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & - & - & 1 & 1 & - & - \\ 1 & 1 & - & - & 1 & 1 & - & - \\ 1 & 1 & - & - & 1 & 1 & - & - \\ 1 & 1 & - & - & 1 & 1 & - & - \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & - & 1 & - & 1 & - & 1 & - \\ 1 & - & 1 & - & 1 & - & 1 & - \\ 1 & - & 1 & - & 1 & - & 1 & - \\ 1 & - & 1 & - & 1 & - & 1 & - \end{pmatrix}$

This basis may be extended to a full basis for 2-cocycles over G_2^t , by simply juxtaposing a basis for 2-coboundaries over G_2^t . Such a basis was described in (8) (according to Lemma 2),

t	basis for 2-coboundaries
$[t]_2 = 1$	$\langle \partial_2, \dots, \partial_{4t-2} \rangle$
$[t]_2 = 0$	$\langle \partial_2, \dots, \partial_{4t-3} \rangle$

Remark 4. All the generators above coincide with those of (8) (notice that G_2^t was denoted G_4^t there), excepting $F^*(e_1^*)$, which is substituted by $\beta_3 = 1_q \otimes BN_{2^r} \otimes 1_4$, for $t = 2^r q$. It may be checked that

$$\beta_3 = F^*(e_1^*) \cdot \prod_{k=0}^{\lfloor \frac{q}{2} \rfloor - 1} \partial_{k2^{r+3}+2^{r+2}+1} \dots \partial_{k2^{r+3}+2^{r+3}}.$$

3.1.3. Basis for $G_3^t = D_{4t} = \mathbf{Z}_2 \times \mathbf{Z}_{2t}$

Notice that the i -th element of G_3 corresponds to $(\lfloor \frac{i-1}{2t} \rfloor, [i-1]_{2t}) \in \mathbf{Z}_2 \times \mathbf{Z}_{2t}$. Conversely, the element $(i_1, i_2) \in \mathbf{Z}_2 \times \mathbf{Z}_{2t}$ corresponds to the i -th element of G_3 , for $i = 2t \cdot i_1 + i_2 + 1$.

In these circumstances, it may be checked that

$$\mathcal{B}_1^* = \{u_1^*, u_2^*\}, \mathcal{B}_2^* = \{e_1^*, e_2^*, e_3^*\}, \mathcal{B}_3^* = \{v_1^*, v_2^*, v_3^*, v_4^*\},$$

$$d^1 = 0, d^2 = 0,$$

$$F[i][j] = [i_1 j_1] \cdot e_1 + [-j_1 (-1)^{i_1} i_2]_{2t} \cdot e_2 + [((-1)^{j_1} j_2]_{2t} + [(-1)^{i_1+j_1} i_2]_{2t} \geq 2t] \cdot e_3 + j_1 [i_2 \geq 1] [((-1)^{i_1+j_1} i_2]_{2t} - 1) \cdot e_3]_2.$$

From these data, it may be checked that

D_1	Q_1^{-1}	D_2	P_2
$0_{2 \times 3}$	I_3	$0_{3 \times 4}$	I_3

Thus $H^2(\text{coh} G_3^t; \mathbf{Z}_2) = \mathbf{Z}_2^3$ and $\mathcal{B}_2^* = \{e_1^*, e_2^*, e_3^*\}$ is a basis for representative 2-cocycles over $\text{coh} G_3^t$. Accordingly, a basis for representative 2-cocycles over G_3^t is given by

$$\{F^*(e_i^*), 1 \leq i \leq 3\}. \text{ More explicitly, } F^*(e_1^*) = BN_2 \otimes 1_{2t}, F^*(e_2^*) = \left(\begin{array}{c|c} 1_{2t} & 1_t \otimes \begin{pmatrix} 1 & 1 \\ - & - \end{pmatrix} \\ \hline 1_{2t} & 1_t \otimes \begin{pmatrix} 1 & 1 \\ - & - \end{pmatrix} \end{array} \right)$$

$$\text{and } F^*(e_3^*) = \left(\begin{array}{c|c} & \begin{pmatrix} 1 & \dots & 1 \\ 1 \\ - \\ BN_{2t} & 1 & B_{2t-1} \\ \vdots \\ - \\ 1 \end{pmatrix} \\ \hline & \begin{pmatrix} 1 \\ 1 \\ - \\ FN_{2t} & 1 & F_{2t-1} \\ \vdots \\ - \\ 1 & - & \dots & - \end{pmatrix} \end{array} \right), \text{ where } FN_k \text{ denotes the forward negacyclic matrix of size } k \times k, FN_k = \begin{pmatrix} 1 & \dots & \dots & 1 \\ 1 & - & \dots & - \\ \vdots & \ddots & \ddots & \vdots \\ 1 & \dots & 1 & - \end{pmatrix}_{k \times k} \text{ and } B_n : \begin{cases} 2k+1 \text{ row} = 2k+1 \text{ row of } -BN_n \\ 2k \text{ row} = 2k \text{ row of } BN_n \end{cases}$$

$$F_n : \begin{cases} 2k+1 \text{ row} = 2k+1 \text{ row of } F_n \\ 2k \text{ row} = 2k \text{ row of } -F_n \end{cases}$$

This basis may be extended to a full basis for 2-cocycles over G_3^t , by simply juxtaposing a basis for 2-coboundaries over G_3^t (see Lemma 2). This basis is $\langle \partial_2, \dots, \partial_{4t-2} \rangle$ as it was pointed out in (8).

Remark 5. A basis $\{\beta_1, \beta_2, \gamma_1\}$ for representative 2-cocycles over G_3^t was already determined in (8) (notice that G_3^t was denoted G_5^t there). It may be checked that $\beta_1 = F^*(e_1^*)$ and $\gamma_1 = F^*(e_2^*) \cdot F^*(e_3^*)$. Depending on the parity of $t > 1$, it may be checked that

$\beta_2 = 1_{2t} \otimes BN_2$ is obtained from the following linear combination

t	combination
$[t]_2 = 1$	$F^*(e_3^*) \cdot \left(\prod_{k=0}^{\lceil \frac{t}{2} \rceil - 1} \partial_{2+4k} \partial_{2+4k} \right) \left(\prod_{k=0}^{\lfloor \frac{t}{2} \rfloor - 1} \partial_{1+4k+4\lceil \frac{t}{2} \rceil} \partial_{2+4k+4\lceil \frac{t}{2} \rceil} \right)$
$[t]_2 = 0$	$F^*(e_2^*) \cdot \left(\prod_{k=0}^{\lceil \frac{t}{2} \rceil - 1} \partial_{2+4k} \partial_{2+4k} \right) \left(\prod_{k=0}^{\lfloor \frac{t}{2} \rfloor - 1} \partial_{1+4k+4\lceil \frac{t}{2} \rceil} \partial_{2+4k+4\lceil \frac{t}{2} \rceil} \right)$

3.2. Constructing basis for 3-cocycles

We now apply the cohomological reduction method for obtaining basis for 3-cocycles over the groups G_1^t and G_2^t .

In what follows, the $n \times n \times n$ -matrix related to a 3-cocycle over a group G of order n will be given as the n horizontal sections $n \times n$ of the 3D-matrix obtained by fixing the third coordinate in all possible ways. As usual, we will use i, j, k for denoting the first, second and third coordinates of a matrix.

3.2.1. Basis for $G_1^t = \mathbf{Z}_{2t} \times \mathbf{Z}_2$

Notice that the i -th element of G_1 corresponds to $(\lfloor \frac{i-1}{2} \rfloor, [i-1]_2) \in \mathbf{Z}_{2t} \times \mathbf{Z}_2$. Conversely, the element $(i_1, i_2) \in \mathbf{Z}_{2t} \times \mathbf{Z}_2$ corresponds to the i -th element of G_1 , for $i = 2i_1 + i_2 + 1$.

In these circumstances, it may be checked that

$$\begin{aligned} \mathcal{B}_2^* &= \{e_1^*, e_2^*, e_3^*\}, \mathcal{B}_3^* = \{v_1^*, v_2^*, v_3^*, v_4^*\}, \mathcal{B}_4^* = \{w_1^*, w_2^*, w_3^*, w_4^*, w_5^*\}, d^2 = 0, d^3 = 0, \\ F[i|j|k] &= [k_1[i_1 + j_1 \geq 2t] \cdot v_1 + k_2[i_1 + j_1 \geq 2t] \cdot v_2 + i_1[j_2 + k_2 \geq 2] \cdot v_3 + \\ &\quad k_2[i_2 + j_2 \geq 2] \cdot v_4]_2. \end{aligned}$$

From these data, it may be checked that

D_2	Q_3^{-1}	D_3	P_3
$0_{3 \times 4}$	I_4	$0_{4 \times 5}$	I_4

Thus $H^3(\text{coh}G_1^t; \mathbf{Z}_2) = \mathbf{Z}_2^4$ and $\mathcal{B}_3^* = \{v_1^*, v_2^*, v_3^*, v_4^*\}$ is a basis for representative 3-cocycles over $\text{coh}G_1^t$. Accordingly, a basis for representative 3-cocycles over G_1^t is given by $\{F^*(v_1^*), F^*(v_2^*), F^*(v_3^*), F^*(v_4^*)\}$.

It may be straightforwardly checked that the $F^*(v_i^*)$ are the 3D-matrices whose horizontal sections are given by:

- $F^*(v_1^*)$: $\{J_{4t}, J_{4t}, BN_{2t} \otimes 1_2, BN_{2t} \otimes 1_2, \dots, J_{4t}, J_{4t}, BN_{2t} \otimes 1_2, BN_{2t} \otimes 1_2\}$.
- $F^*(v_2^*)$: $\{J_{4t}, BN_{2t} \otimes 1_2, \dots, J_{4t}, BN_{2t} \otimes 1_2\}$.

$$\bullet F^*(v_3^*): \{J_{4t}, 1_t \otimes A, \dots, J_{4t}, 1_t \otimes A\}, \text{ for } A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & - & 1 & - \\ 1 & - & 1 & - \end{pmatrix}.$$

- $F^*(v_4^*)$: $\{J_{4t}, 1_{2t} \otimes BN_2, \dots, J_{4t}, 1_{2t} \otimes BN_2\}$.

This basis may be extended to a full basis for 3-cocycles over G_1^t , by simply juxtaposing a basis for 3-coboundaries over G_1^t (see Lemma 2). This basis is given by $\langle \partial_1, \dots, \partial_{16t^2-4t-2}, \partial_{16t^2-4t+1} \rangle$.

3.2.2. Basis for $G_2^t = \mathbf{Z}_t \times \mathbf{Z}_2^2 = \mathbf{Z}_t \times (\mathbf{Z}_2 \times \mathbf{Z}_2)$

Notice that the i -th element of G_2 corresponds to $(\lfloor \frac{i-1}{4} \rfloor, \lfloor \lfloor \frac{i-1}{2} \rfloor \rfloor_2, [i-1]_2) \in \mathbf{Z}_t \times (\mathbf{Z}_2 \times \mathbf{Z}_2)$. Conversely, the element $(i_1, i_2, i_3) \in \mathbf{Z}_t \times (\mathbf{Z}_2 \times \mathbf{Z}_2)$ corresponds to the i -th element of G_2 , for $i = 4i_1 + 2i_2 + i_3 + 1$.

In these circumstances, it may be checked that

$$\begin{aligned} \mathcal{B}_2^* &= \{e_1^*, \dots, e_6^*\}, \mathcal{B}_3^* = \{v_1^*, \dots, v_{10}^*\}, \mathcal{B}_4^* = \{w_1^*, \dots, w_{15}^*\}, \\ d^2(e_2^*) &= t \cdot v_2^*, d^2(e_3^*) = t \cdot v_3^*, d^3(v_1^*) = tw_1^*, d^3(v_4^*) = tw_4^*, d^3(v_5^*) = tw_5^*, d^3(v_6^*) = tw_6^*, \\ F[i, j, k] &= [k_1[i_1 + j_2 \geq t] \cdot v_1 + k_2[i_1 + j_1 \geq t] \cdot v_2 + \\ &k_3[i_1 + j_1 \geq t] \cdot v_3 + i_1[j_2 + k_2 \geq 2] \cdot v_4 + i_1 j_2 k_3 \cdot v_5 + i_1[j_3 + k_3 \geq 2] \cdot v_6 + k_2[i_2 + j_2 \geq 2] \cdot v_7 + \\ &k_3[i_2 + j_2 \geq 2] \cdot v_8 + i_2[j_3 + k_3 \geq 2] \cdot v_9 + k_3[i_3 + j_3 \geq 2] \cdot v_{10}]_2. \end{aligned}$$

From these data, it may be checked that:

t	D_2	Q_2^{-1}	D_3	P_3
$[t]_2 = 0$	$0_{6 \times 10}$	I_{10}	$0_{10 \times 15}$	I_{10}
$[t]_2 = 1$	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

$$\text{Thus } H^3(\text{coh}G_2^t; \mathbf{Z}_2) = \begin{cases} \mathbf{Z}_2^3 & \text{for } t \text{ odd} \\ \mathbf{Z}_2^{10} & \text{for } t \text{ even} \end{cases} \text{ and } \begin{cases} \{v_7^*, v_8^*, v_9^*, v_{10}^*\} & \text{for } t \text{ odd} \\ \{v_1^*, \dots, v_{10}^*\} & \text{for } t \text{ even} \end{cases} \text{ is a basis}$$

for representative 3-cocycles over $\text{coh}G_2^t$. Accordingly, a basis for representative 3-cocycles over G_2^t consists in the correspondent $F^*(v_i^*)$.

It may be straightforwardly checked that the $F^*(v_i^*)$ are the 3D-matrices whose horizontal sections are given by:

- $F^*(v_1^*)$: $\{J_{4t}, J_{4t}, J_{4t}, J_{4t}, BN_t \otimes 1_4, BN_t \otimes 1_4, BN_t \otimes 1_4, BN_t \otimes 1_4, \dots, J_{4t}, J_{4t}, J_{4t}, J_{4t}, BN_t \otimes 1_4, BN_t \otimes 1_4, BN_t \otimes 1_4, BN_t \otimes 1_4\}$.
- $F^*(v_2^*)$: $\{J_{4t}, J_{4t}, BN_t \otimes 1_4, BN_t \otimes 1_4, \dots, J_{4t}, J_{4t}, BN_t \otimes 1_4, BN_t \otimes 1_4\}$.
- $F^*(v_3^*)$: $\{J_{4t}, BN_t \otimes 1_4, \dots, J_{4t}, BN_t \otimes 1_4\}$.

- $F^*(v_4^*)$: $\{J_{4t}, J_{4t}, 1_{\frac{t}{2}} \otimes A \otimes 1_2, 1_{\frac{t}{2}} \otimes A \otimes 1_2, \dots, J_{4t}, J_{4t}, 1_{\frac{t}{2}} \otimes A \otimes 1_2, 1_{\frac{t}{2}} \otimes A \otimes 1_2\}$, for

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & - \\ 1 & -1 & -1 & - \end{pmatrix}.$$

- $F^*(v_5^*)$: $\{J_{4t}, 1_{\frac{t}{2}} \otimes A \otimes 1_2, \dots, J_{4t}, 1_{\frac{t}{2}} \otimes A \otimes 1_2\}$.

- $F^*(v_6^*)$: $\{J_{4t}, 1_{\frac{t}{2}} \otimes \begin{pmatrix} J_4 & J_4 \\ B & B \end{pmatrix}, \dots, J_{4t}, 1_{\frac{t}{2}} \otimes \begin{pmatrix} J_4 & J_4 \\ B & B \end{pmatrix}\}$, for $B = \begin{pmatrix} 1 & -1 & - \\ 1 & -1 & - \\ 1 & -1 & - \\ 1 & -1 & - \end{pmatrix}$.

- $F^*(v_7^*)$: $\{J_{4t}, J_{4t}, 1_t \otimes BN_2 \otimes 1_2, 1_t \otimes BN_2 \otimes 1_2, \dots, J_{4t}, J_{4t}, 1_t \otimes BN_2 \otimes 1_2, 1_t \otimes BN_2 \otimes 1_2\}$.
- $F^*(v_8^*)$: $\{J_{4t}, 1_t \otimes BN_2 \otimes 1_2, \dots, J_{4t}, 1_t \otimes BN_2 \otimes 1_2\}$.
- $F^*(v_9^*)$: $\{J_{4t}, 1_t \otimes A, \dots, J_{4t}, 1_t \otimes A\}$.
- $F^*(v_{10}^*)$: $\{J_{4t}, 1_{2t} \otimes BN_2, \dots, J_{4t}, 1_{2t} \otimes BN_2\}$.

This basis may be extended to a full basis for 3-cocycles over G_2^t , by simply juxtaposing a basis for 3-coboundaries over G_2^t . According to Lemma 2, such a basis is given by

t	basis for 3-coboundaries
$[t]_2 = 1$	$\langle \partial_1, \dots, \partial_{16t^2-4t-2}, \partial_{16t^2-4t+1} \rangle$
$[t]_2 = 0$	$\langle \partial_1, \dots, \partial_{16t^2-8t-3}, \partial_{16t^2-8t+1}, \dots, \partial_{16t^2-4t-3}, \partial_{16t^2-4t+1}, \dots, \partial_{16t^2-4t+3} \rangle$

4. Conclusions and further work

The cohomological reduction method has been shown to produce a basis for 2-cocycles over finite groups for which a cohomological model is known. This procedure is more elegant and economical (from a computational point of view) than any other method known so far. In addition, it may be used to construct a basis for higher dimensional cocycles.

At this point, extending the 2-dimensional case, the n -dimensional cocyclic matrices could be a source for higher dimensional proper (and even improper) Hadamard matrices. As introduced by Shlichta in (18; 19), an *improper* n -dimensional Hadamard matrix of order v is a (± 1) array $A = (a(i_1, i_2, \dots, i_n))_{1 \leq i_j \leq v, 1 \leq j \leq n}$ such that all its parallel $(n-1)$ -dimensional sections are mutually orthogonal; that is, for each $1 \leq l \leq n$, and for all indices x and y in dimension l ,

$$\sum_{j \neq l} \sum_{1 \leq i_j \leq v} a(i_1, \dots, x, \dots, i_j, \dots, i_n) \cdot a(i_1, \dots, y, \dots, i_j, \dots, i_n) = v^{n-1} \delta_{xy}.$$

An improper n -dimensional Hadamard matrix may have stronger orthogonality properties in some dimensions. For instance, an n -dimensional Hadamard matrix A is termed *proper* (see (15; 21) for details) if any parallel rows in A are orthogonal; that is, for each

pair of dimensions j, l , for all indices x and y in dimension l , and for each set of fixed indices in the other $n - 2$ dimensions,

$$\sum_{1 \leq i_j \leq v} a(i_1, \dots, x, \dots, i_j, \dots, i_n) \cdot a(i_1, \dots, y, \dots, i_j, \dots, i_n) = v \delta_{xy}.$$

It follows that every planar section of a proper n -dimensional Hadamard matrix of order v is a Hadamard matrix of order v itself, so v must be 2 or a multiple of 4. Actually, four direct constructions of proper higher dimensional Hadamard matrices are known, depending respectively on the existence of special Boolean functions, planar Hadamard matrices, group developed Hadamard matrices over certain abelian group, and n -dimensional perfect arrays $PBA(r, \dots, r)$ of energy r^n (see (15) for details). In particular, if the Hadamard Conjecture is true, there would exist proper n -dimensional Hadamard matrices for every $n \geq 2$ and every v multiple of 4.

On the contrary, few is known about the existence of improper n -dimensional Hadamard matrices of order v . It is evident that v must be even, but surprisingly it need not to be a multiple of 4. There are well known examples of improper 3, 4-dimensional Hadamard matrices of order 6 (see (21) for instance). Anyway, it is not known whether improper n -dimensional Hadamard matrices exist for all even orders $v = 2t$, $n \geq 3$. Furthermore, only 3-dimensional improper Hadamard matrices of order $2 \cdot 3^b$ are known, $b \geq 1$.

Since the cocyclic framework has showed to be a promising technique for handling with planar Hadamard matrices, we wonder if n -dimensional cocyclic matrices of order v might be suitable as well for looking for n -dimensional improper Hadamard matrices of order v .

For instance, an exhaustive computer search for 3-dimensional Hadamard matrices of order 4 cocyclic over $G_1^1 = G_2^1$ (here $t = 1$), yields that there are 64 improper 3-dimensional Hadamard matrices, none of which is in addition proper. For instance, the matrix related to the product of 3-coboundaries $\partial_4 \partial_7 \partial_{10} \partial_{13}$, whose horizontal sections

are given by $\left\{ \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & - & - \\ 1 & - & 1 & - \\ - & 1 & 1 & - \end{pmatrix}, \begin{pmatrix} 1 & 1 & - & - \\ 1 & 1 & 1 & 1 \\ 1 & - & - & 1 \\ - & 1 & - & 1 \end{pmatrix}, \begin{pmatrix} 1 & - & 1 & - \\ 1 & - & - & 1 \\ 1 & 1 & 1 & 1 \\ - & - & 1 & 1 \end{pmatrix}, \begin{pmatrix} - & 1 & 1 & - \\ - & 1 & - & 1 \\ - & - & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \right\}.$

Since there is more interest in improper n -dimensional Hadamard matrices of even order $v = 4k + 2$ (in particular, for $n = 3$ and $v \neq 2 \cdot 3^b$), it would be necessary to fix a group G different from those described in this paper (all of which have order a multiple of 4).

For instance, take $G^t = \mathbf{Z}_{2t}$, t odd. We now apply the cohomological reduction method for calculating a basis for n -cocycles over G . It may be checked that $\mathcal{B}_2^* = \{e_1^*\}$, $\mathcal{B}_3^* = \{v_1^*\}$, $\mathcal{B}_4^* = \{w_1^*\}$, $d^2 = 0$, $d^3 = 0$, $F[i|j|k] = [k[i + j \geq 2t]]_2 \cdot v_1$.

From these data, it may be checked that

D_2	Q_3^{-1}	D_3	P_3
0	1	0	1

Thus $H^3(\text{coh}G^t; \mathbf{Z}_2) = \mathbf{Z}_2$ and $\mathcal{B}_3^* = \{v_1^*\}$ is a basis for representative 3-cocycles over $\text{coh}G^t$. Accordingly, a basis for representative 3-cocycles over G^t is given by $\{F^*(v_1^*)\}$, whose horizontal sections are given by $\{J_{2t}, BN_{2t}, \dots, J_{4t}, BN_{2t}\}$.

This basis may be extended to a full basis for 3-cocycles over G^t , by simply juxtaposing a basis for 3-coboundaries over G^t (see Lemma 2). This basis is given by $\langle \partial_1, \dots, \partial_{4t^2-2t} \rangle$.

An exhaustive computer search for 3-dimensional Hadamard matrices of order 4 cocyclic over G^2 (i.e. G^t for $t = 2$), yields that there are 32 improper 3-dimensional Hadamard matrices, none of which is in addition proper. For instance, the matrix related to the product of 3-coboundaries $\partial_4 \partial_7 \partial_8 \partial_9$, whose horizontal sections are given by

$$\left\{ \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & - & - & 1 \\ - & 1 & - & 1 \\ 1 & 1 & - & - \end{pmatrix}, \begin{pmatrix} 1 & 1 & - & - \\ 1 & - & 1 & - \\ - & 1 & 1 & - \\ 1 & 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & - & 1 & - \\ 1 & 1 & - & - \\ - & - & - & - \\ 1 & - & - & 1 \end{pmatrix}, \begin{pmatrix} - & 1 & 1 & - \\ - & - & - & - \\ 1 & 1 & - & - \\ - & 1 & - & 1 \end{pmatrix} \right\}.$$

Unfortunately, the computer could not afford calculations for improper 3-dimensional \mathbf{Z}_{2t} -cocyclic Hadamard matrices of order 10 (i.e. $t = 5$). Contributions in this sense (with this or other groups of order $v = 4k + 2$) would be appreciated, since improper 3-dimensional Hadamard matrices of order different to $2 \cdot 3^b$ are still to be discovered.

Finally, we would like to conclude this paper proposing several problems concerning higher dimensional (cocyclic) Hadamard matrices, which should be studied in a near future, such as:

Problem 1. Characterize n -dimensional proper and improper cocyclic Hadamard matrices.

One could think that an n -dimensional cocyclic matrix would consist of cocyclic layers. This is not true at all, as it will be discussed elsewhere. Thus the traditional cocyclic test for 2-dimensional cocyclic Hadamard matrices cannot be naturally extended to the n -dimensional case so far. A deeper analysis must be done.

Problem 2. Look for n -dimensional proper and improper cocyclic Hadamard matrices.

This is a very difficult task, since the search space seems to grow drastically in exponential size (e.g., the basis for 3-dimensional cocyclic matrices over G_1^t and G_2^t , t even, consists of $16t^2 - 4t + 3$ and $16t^2 - 4t + 7$ generators, respectively).

Problem 3. Determine whether any of the already known construction methods for generating higher dimensional proper Hadamard matrices from 2-dimensional ones (see (16) for details) involves n -cocyclic matrices.

Furthermore, is it possible to derive a method for constructing n -dimensional proper/improper cocyclic Hadamard matrices from lower dimensional proper/improper cocyclic Hadamard matrices (e.g. via the cup product in cohomology)?

Problem 4. Does an n -dimensional improper cocyclic Hadamard conjecture make sense?

Proper higher dimensional Hadamard matrices might exist only for orders 1,2 and a multiple of 4, and they do exist if and only if a planar Hadamard matrix of the same order do exist. This reduces the problem to the Hadamard Conjecture. Nevertheless, as it has been discussed earlier, only higher dimensional improper Hadamard matrices of order $2 \cdot 3^b$ are known. So the question is widely open for the improper case.

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